SPHERICAL TRIGONOMETRY.

69. Spherical Trigonometry is that branch of Mathematics which treats of the solution of spherical triangles.

In every spherical triangle there are six parts: three sides and three angles. In general, any three of these parts being given, the remaining parts may be found.

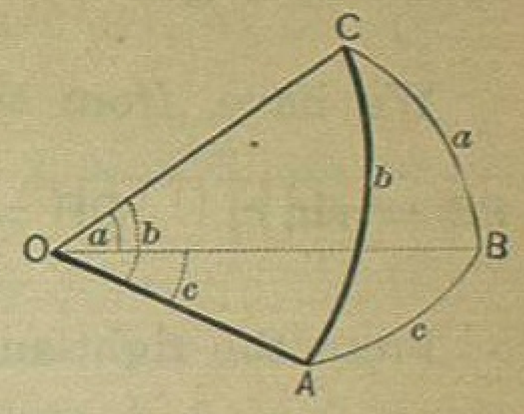
GENERAL PRINCIPLES.

70. For the purpose of deducing the formulas required in the solution of spherical triangles, we shall suppose the triangles to be situated on spheres whose radii are equal to 1. The formulas thus deduced may be rendered applicable to triangles lying on any sphere, by making them homogeneous in terms of the radius of that sphere, as explained in Art. 30. The only cases considered will be those in which each of the sides and angles is less than 180°.

Any angle of a spherical triangle is the same as the diedral angle included by the planes of its sides, and its measure is equal to that of the angle included between two right lines, one in each plane, and both perpendicular to their common intersection at the same point (B. VI., D. 4).

The radius of the sphere being equal to 1, each side of the triangle will measure the angle, at the centre, subtended by it. Thus, in the triangle ABC, the angle at A

is the same as that included between the planes AOC and AOB; and the side a is the measure of the plane angle BOC, O being the centre of the sphere, and OB the radius, equal to 1.



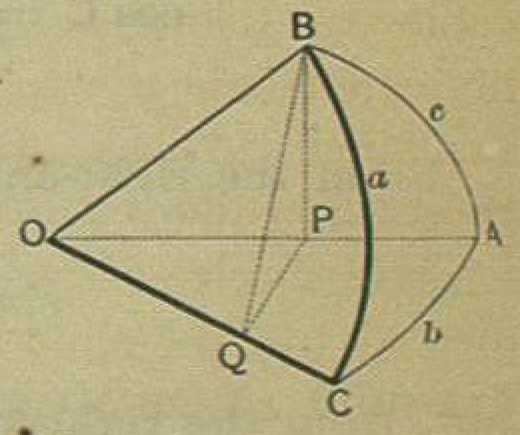
71. Spherical triangles, like plane triangles, are divided into two classes, *right-angled spherical triangles*, and *oblique-angled spherical triangles.* Each class will be considered in turn.

We shall, as before, denote the angles by the capital letters A, B, and C, and the sides opposite by the small letters a, b, and c.

FORMULAS

USED IN SOLVING RIGHT-ANGLED SPHERICAL TRIANGLES.

72. Let CAB be a sperical triangle, right-angled at A, and let O be the centre of the sphere on which it is situated. Denote the angles of the triangle by the letters A, B, and C, and the sides opposite by the letters a, b, and c, recollecting that B and C may change places, provided that b and c change places at the same time.



Draw OA, OB, and OC, each equal to 1. From B, draw BP perpendicular to OA, and from P draw PQ perpendicular to OC; then join the points Q and B, by the line QB. The line QB will be perpendicular to OC (B. VI., P. VI.), and the angle PQB will be equal to the inclination of the

planes OCB and OCA; that is, it will be equal to the spherical angle C.

We have, from the figure,

$$\text{PB} = \sin c, \qquad \text{OP} = \cos c, \qquad \text{QB} = \sin a, \qquad \text{OQ} = \cos a.$$

From the right-angled triangles OQP and QPB, we have

$$\text{OQ} = \text{OP} \cos \text{AOC}; \quad \text{or,} \quad \cos a = \cos c \cos b. \quad \cdot \quad (1.)$$

$$\text{PB} = \text{QB} \sin \text{PQB}; \quad \text{or,} \quad \sin c = \sin a \sin \text{C}. \quad \cdot \quad (2.)$$

From the right-angled triangle QPB, we have

$$\cos \text{PQB}, \text{ or } \cos \text{C} = \frac{\text{QP}}{\text{QB}};$$

but, from the right-angled triangle PQO, we have

$$\text{QP} = \text{OQ} \tan \text{QOP} = \cos a \tan b;$$

substituting for QP and QB their values, we have

$$\cos \text{C} = \frac{\cos a \tan b}{\sin a} = \cot a \tan b. \quad \cdot \cdot \quad (3.)$$

From the right-angled triangle OQP, we have

$$\sin \text{QOP}, \text{ or } \sin b = \frac{\text{QP}}{\text{OP}};$$

but, from the right-angled triangle QPB, we have

$$\text{QP} = \text{PB} \cot \text{PQB} = \sin c \cot \text{C};$$

substituting for QP and OP their values, we have

$$\sin b = \frac{\sin c \cot \text{C}}{\cos c} = \tan c \cot \text{C}. \quad \cdot \cdot \cdot \quad (4.)$$

If, in (2), we change c and C into b and B, we have

$$\sin b = \sin a \sin \text{B}. \quad \cdot \cdot \cdot \cdot \cdot \quad (5.)$$

If, in (3), we change b and C into c and B, we have

$$\cos \text{B} = \cot a \tan c. \quad \cdot \cdot \cdot \cdot \cdot \quad (6.)$$

If, in (4), we change b, c, and C, into c, b, and B, we have

$$\sin c = \tan b \cot \text{B}. \quad \cdot \cdot \cdot \cdot \cdot \quad (7.)$$

Multiplying (4) by (7), member by member, we have

$$\sin b \sin c = \tan b \tan c \cot \text{B} \cot \text{C}.$$

Dividing both members by $\tan b \tan c$, we have

$$\cos b \cos c = \cot \text{B} \cot \text{C};$$

and substituting for $\cos b \cos c$, its value, $\cos a$, taken from (1), we have

$$\cos a = \cot \text{B} \cot \text{C}. \quad \cdot \cdot \cdot \cdot \cdot \quad (8.)$$

Formula (6) may be written under the form

$$\cos \text{B} = \frac{\cos a \sin c}{\sin a \cos c}.$$

Substituting for $\cos a$, its value, $\cos b \cos c$, taken from (1), and reducing, we have

$$\cos \text{B} = \frac{\cos b \sin c}{\sin a}.$$

Again, substituting for $\sin c$, its value, $\sin a \sin \text{C}$, taken from (2), and reducing, we have

$$\cos \mathrm{B} = \cos b \sin \mathrm{C}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9.)$$

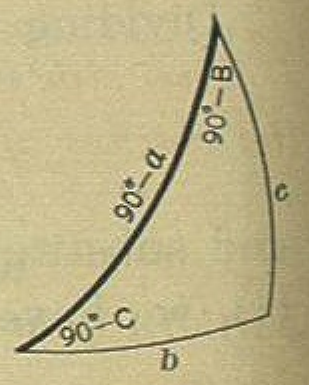
Changing B, b, and C, in (9), into C, c, and B, we have

$$\cos \mathrm{C} = \cos c \sin \mathrm{B}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; (10.)$$

These ten formulas are sufficient for the solution of any right-angled spherical triangle whatever. For the purpose of classifying them under two general rules, and for convenience in remembering them, these formulas are usually put under other forms by the use of

NAPIER'S CIRCULAR PARTS.

73. *The two sides about the right angle, the complements of their opposite angles, and the complement of the hypothenuse,* are called Napier's Circular Parts.



If we take *any three* of the five parts, as shown in the figure, they will either be *adjacent* to each other, or one of them will be separated from each of the two others by an intervening part. In the first case, the one lying between the two other parts is called the *middle part,* and the two others, *adjacent parts.* In the second case, the one separated from both the other parts, is called the *middle part,* and the two others, *opposite parts.* Thus, if $90° - a$ is the middle part, $90° - \mathrm{B}$ and $90° - \mathrm{C}$ are *adjacent parts;* and b and c are *opposite parts;* if c is the middle part, b and $90° - \mathrm{B}$ are *adjacent parts* (the right angle not being considered), and $90° - \mathrm{C}$ and $90° - a$ are *opposite parts:* and similarly, for each of the other parts, taken as a middle part.

74. Let us now consider, in succession, each of the five parts as a middle part, when the two other parts are opposite. Beginning with the hypothenuse, we have, from formulas (1), (2), (5), (9), and (10), Art. 72,

$$\sin (90° - a) = \cos b \cos c; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1.)$$

$$\sin c = \cos (90° - a) \cos (90° - \mathrm{C}); \quad (2.)$$

$$\sin b = \cos (90° - a) \cos (90° - \mathrm{B}); \quad (3.)$$

$$\sin (90° - \mathrm{B}) = \cos b \cos (90° - \mathrm{C}); \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (4.)$$

$$\sin (90° - \mathrm{C}) = \cos c \cos (90° - \mathrm{B}). \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (5.)$$

Comparing these formulas with the figure, we see that

The sine of the middle part is equal to the rectangle of the cosines of the opposite parts.

Let us now take the same middle parts, and the other parts adjacent. Formulas (8), (7), (4), (6), and (3), Art. 72, give

$$\sin (90° - a) = \tan (90° - \mathrm{B}) \tan (90° - \mathrm{C}); \quad (6.)$$

$$\sin c = \tan b \tan (90° - \mathrm{B}); \quad \cdot \quad \cdot \quad \cdot \quad (7.)$$

$$\sin b = \tan c \tan (90° - \mathrm{C}); \quad \cdot \quad \cdot \quad \cdot \quad (8.)$$

$$\sin (90° - \mathrm{B}) = \tan (90° - a) \tan c; \quad \cdot \quad \cdot \quad \cdot \quad (9.)$$

$$\sin (90° - \mathrm{C}) = \tan (90° - a) \tan b. \quad \cdot \quad \cdot \quad \cdot \; (10.)$$

Comparing these formulas with the figure, we see that

The sine of the middle part is equal to the rectangle of the tangents of the adjacent parts.

These two rules are called Napier's rules for circular parts, and are sufficient to solve any right-angled spherical triangle.

75. In applying Napier's rules for circular parts, the part sought will be determined by its sine. Now, the same sine corresponds to two different arcs, or angles, supplements of each other; it is, therefore, necessary to discover such relations between the given and the required parts, as will serve to point out which of the two arcs, or angles, is to be taken.

Two parts of a spherical triangle are said to be of *the same species*, when they are each less than 90°, or each greater than 90°; and of *different species*, when one is less and the other greater than 90°.

From formulas (9) and (10), Art. 72, we have,

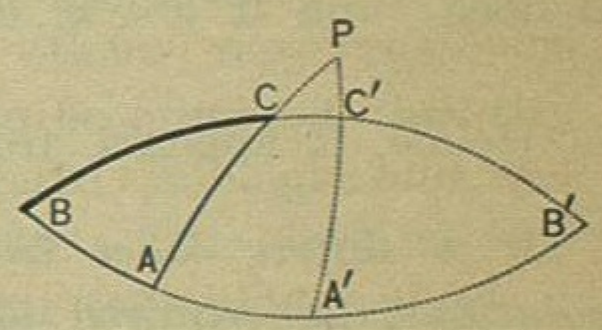
$$\sin C = \frac{\cos B}{\cos b}, \quad \text{and} \quad \sin B = \frac{\cos C}{\cos c};$$

since the angles B and C are each less than 180°, their sines must always be positive: hence, cos B must have the same sign as cos b, and the cos C must have the same sign as cos c. This can only be the case when B is of the same species as b, and C of the same species as c; that is, *each side about the right angle is always of the same species as its opposite angle.*

From formula (1), we see that when a is less than 90°, or when cos a is positive, the cosines of b and c will have the same sign; and hence, b and c will be of the *same species:* when a is greater than 90°, or when cos a is negative, the cosines of b and c will have contrary signs, and hence b and c will be of *different species:*

therefore, *when the hypothenuse is less than 90°, the two sides about the right angle, and consequently the two oblique angles, will be of the same species; when the hypothenuse is greater than 90°, the two sides about the right angle, and consequently the two oblique angles, will be of different species.*

These two principles enable us to determine the nature of the part sought, in every case, except when an oblique angle and the side opposite are given, to find the remaining parts. In this case, there may be *two solutions, one solution*, or *no solution.*

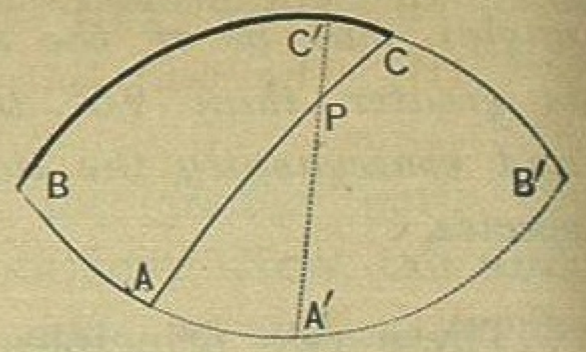


There may be two cases:

1°. Let there be given B and b, and B *acute.* Construct B and prolong its sides till they meet in B′. Then will BCB′ and BAB′ be semi-circumferences of great circles, and the spherical angles B and B′ will be equal to each other. As B is acute, its measure is the *longest* arc of a great circle that can be drawn perpendicular to the side BA and included between the sides of the angle B (B. IX., Gen. S. 2); hence, if the given side is *greater* than the measure of the given angle opposite, that is, if $b >$ B, no triangle can be constructed, that is, there can be *no solution:* if $b =$ B, BC′ and BA′ will each be a quadrant (B. IX., P. IV.), and the triangle BA′C′, or its equal B′A′C′, will be birectangular (B. IX., P. XIV., C. 3), and there will be but *one solution:* if $b <$ B, there will be *two solutions*, BAC and B′AC, the required parts of one being supplements of the required parts of the other.

Since B < 90°, if $b <$ B, b differs *more* from 90° than B does; and if $b >$ B, b differs *less* from 90° than B.

2d. Let B be *obtuse*. Construct B as before. As B is obtuse, its measure is the *shortest* arc of a great circle that can be drawn perpendicular to the side BA and included between the sides of the angle B (B. IX., Gen. S. 2); hence, if $b <$ B, there can be *no solution:* if $b =$ B, the corresponding triangle, BA′C′ or B′A′C′, will be birectangular and there will be but *one solution*, as before: and if $b >$ B, there will be *two solutions*, BAC and B′AC.



Since B $> 90°$, if $b >$ B, b differs *more* from $90°$ than B does; and if $b <$ B, b differs *less* from $90°$ than B.

Hence, it appears, from both cases, that

If b differs *more* from $90°$ than B, there will be *two solutions,* the required parts in the one case being supplements of the required parts in the other case.

If $b =$ B, the triangle will be birectangular, and there will be but *one solution.*

If b differs *less* from $90°$ than B, the triangle can not be constructed, that is, there will be *no solution.*

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

76. In a right-angled spherical triangle, the right angle is always known. If any two of the other parts are given, the remaining parts may be found by Napier's rules for circular parts. Six cases may arise. There may be given,

I. The hypothenuse and one side.

II. The hypothenuse and one oblique angle.

III. The two sides about the right angle.

IV. One side and its adjacent angle.

V. One side and its opposite angle.

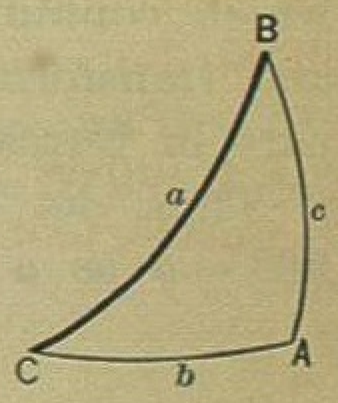
VI. The two oblique angles.

In any one of these cases, we select that part which is either adjacent to, or separated from, each of the other given parts, and calling it the middle part, we employ that one of Napier's rules which is applicable. Having determined a third part, the two others may then be found in a similar manner. It is to be observed, that the formulas employed are to be rendered homogeneous, in terms of R, as explained in Art. 30. This is done by simply multiplying the radius, R, into the middle part.

Examples.

1. Given $a = 105° \, 17' \, 29''$, and $b = 38° \, 47' \, 11''$, to find C, c, and B.

Since $a > 90°$, b and c must be of different species, that is, $c > 90°$, and hence C $> 90°$.



Operation.

Formula (10), Art. 74, gives for $90° -$ C, middle part,

$$\log \cos C = \log \cot a + \log \tan b - 10;$$

log cot a	$(105° \, 17' \, 29'')$	9.436811	
log tan b	$(38° \, 47' \, 11'')$	9.905055	
log cos C	. . .	9.341866	$\therefore$ C $= 102° \, 41' \, 33''$.

Formula (2), Art. 74, gives for c, middle part,

$$\log \sin c = \log \sin a + \log \sin C - 10;$$

$$\begin{array}{lll} \log \sin a \ (105^\circ\, 17'\, 29'') & 9.984346 & \\ \log \sin C \ (102^\circ\, 41'\, 33'') & \underline{9.989256} & \\ \qquad \log \sin c \quad \cdot \ \cdot \ \cdot & \underline{9.973602} & \therefore\ c = 109^\circ\, 46'\, 32''. \end{array}$$

Formula (4) gives for $90^\circ - B$, middle part,

$$\log \cos B = \log \sin C + \log \cos b - 10;$$

$$\begin{array}{lll} \log \sin C \ (102^\circ\, 41'\, 33'') & 9.989256 & \\ \log \cos b \ \ (38^\circ\, 47'\, 11'') & \underline{9.891808} & \\ \qquad \log \cos B \quad \cdot \ \cdot \ \cdot & \underline{9.881064} & \therefore\ B = 40^\circ\, 29'\, 50''. \end{array}$$

Ans. $c = 109^\circ\, 46'\, 32''$, $B = 40^\circ\, 29'\, 50''$, $C = 102^\circ\, 41'\, 33''$.

It is better, in all cases, to find the required parts in terms of the two given parts. This may always be done by one of the formulas of Art. 74. Select the formula which contains the two given parts and the required part, and transform it, if necessary, so as to find the required part in terms of the given parts.

Thus, let a and B be given, to find C. Regarding $90^\circ - a$ as a middle part, we have, from formula (6),

$$\cos a = \cot B \cot C;$$

whence,
$$\cot C = \frac{\cos a}{\cot B};$$

and, by the application of logarithms,

$$\log \cot C = \log \cos a + (\text{a. c.}) \log \cot B;$$

from which C may be found. In like manner, other cases may be treated.

2. Given $b = 51^\circ\, 30'$, and $B = 58^\circ\, 35'$, to find a, c, and C.

Because $b < B$, there are two solutions.

Operation.

Formula (7) gives for c, middle part,

$$\log \sin c = \log \tan b + \log \cot B - 10;$$

$$\begin{array}{lll} \log \tan b \ \ (51^\circ\, 30') & 10.099395 & \\ \log \cot B \ \ (58^\circ\, 35') & \underline{\ 9.785900} & \\ \qquad \log \sin c \quad \cdot \ \cdot \ \cdot & \underline{\ 9.885295} & \therefore\ c = \ \ 50^\circ\, 09'\, 51'', \\ & & \text{and} \ \ c' = 129^\circ\, 50'\, 09''. \end{array}$$

Formula (3) gives

$$\sin b = \sin a \sin B,$$

whence,
$$\sin a = \frac{\sin b}{\sin B},$$

and hence,
$$\log \sin a = \log \sin b + (\text{a. c.}) \log \sin B;$$

$$\begin{array}{lll} \log \sin b \ \ (51^\circ\, 30') & 9.893544 & \\ (\text{a. c.}) \log \sin B \ \ (58^\circ\, 35') & \underline{0.068848} & \\ \qquad \log \sin a \quad \cdot \ \cdot & 9.962392 & \therefore\ a = \ \ 66^\circ\, 29'\, 53'', \\ & & a' = 113^\circ\, 30'\, 07''. \end{array}$$

Formula (4) gives

$$\cos B = \cos b \sin C,$$

whence,
$$\sin C = \frac{\cos B}{\cos b},$$

and hence,
$$\log \sin C = \log \cos B + (\text{a. c.}) \log \cos b;$$

$$\begin{array}{lll} \log \cos B \ \ (58^\circ\, 35') & 9.717053 & \\ (\text{a. c.}) \log \cos b \ \ (51^\circ\, 30') & \underline{0.205850} & \\ \qquad \log \sin C \quad \cdot \ \cdot & 9.922903 & \therefore\ C = \ \ 56^\circ\, 51'\, 38'', \\ & & C' = 123^\circ\, 08'\, 22''. \end{array}$$

As a *check*, to test the accuracy of the above work, formula (2) may be used. Thus, from that formula,

$$\log \sin c = \log \sin a + \log \sin C - 10.$$

As found above,

$\log \sin a$	. .	9.962392
$\log \sin C$	. .	9.922903
$\log \sin c$	.	9.885295

As the test is satisfied, the work is probably correct. Other cases may be treated in like manner.

3. Given $a = 86° \, 51'$, and $B = 18° \, 03' \, 32''$, to find b, c, and C.

Ans. $b = 18° \, 01' \, 50''$, $c = 86° \, 41' \, 14''$, $C = 88° \, 58' \, 25''$.

4. Given $b = 155° \, 27' \, 54''$, and $c = 29° \, 46' \, 08''$, to find a, B, and C.

Ans. $a = 142° \, 09' \, 13''$, $B = 137° \, 24' \, 21''$, $C = 54° \, 01' \, 16''$.

5. Given $c = 73° \, 41' \, 35''$, and $B = 99° \, 17' \, 33''$, to find a, b, and C.

Ans. $a = 92° \, 42' \, 17''$, $b = 99° \, 40' \, 30''$, $C = 73° \, 54' \, 47''$.

6. Given $b = 115° \, 20'$, and $B = 91° \, 01' \, 47''$, to find a, c, and C.

$a = 64° \, 41' \, 11''$, $c = 177° \, 49' \, 27''$, $C = 177° \, 35' \, 36''$.

$a' = 115° \, 18' \, 49''$, $c' = 2° \, 10' \, 33''$, $C' = 2° \, 24' \, 24''$.

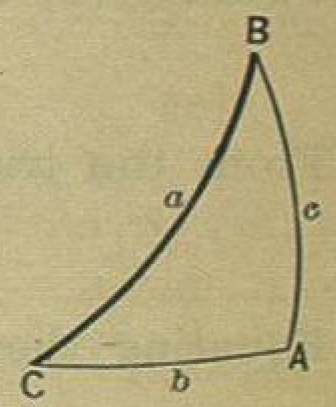
7. Given $B = 47° \, 13' \, 43''$, and $C = 126° \, 40' \, 24''$, to find a, b, and c.

Ans. $a = 133° \, 32' \, 26''$, $b = 32° \, 08' \, 56''$, $c = 144° \, 27' \, 03''$.

QUADRANTAL SPHERICAL TRIANGLES.

77. A Quadrantal Spherical Triangle is one in which one side is equal to 90°. To solve such a triangle, we pass to its supplemental polar triangle, by subtracting each side and each angle from 180° (B. IX., P. VI.). The resulting polar triangle will be right-angled, and may be solved by the rules already given. The supplemental polar triangle of any quadrantal triangle being solved, the parts of the given triangle may be found by subtracting each part of the supplemental triangle from 180°.

Example.



Let A′B′C′ be a quadrantal triangle, in which

$$B'C' = 90°,$$

$$B' = 75° \, 42',$$

and

$$c' = 18° \, 37'.$$

Passing to the supplemental polar triangle, we have

$$A = 90°, \quad b = 104° \, 18', \quad \text{and} \quad C = 161° \, 23'.$$

Solving this triangle by previous rules, we find

$$a = 76° \, 25' \, 11'', \quad c = 161° \, 55' \, 20'', \quad B = 94° \, 31' \, 21'';$$

hence, the required parts of the given quadrantal triangle are,

$$A' = 103° \, 34' \, 49'', \quad C' = 18° \, 04' \, 40'', \quad b' = 85° \, 28' \, 39'',$$

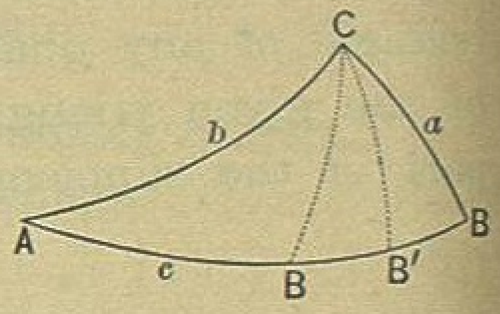
Other quadrantal triangles may be solved in like manner.

FORMULAS

USED IN SOLVING OBLIQUE-ANGLED SPHERICAL TRIANGLES.

78. To show that, in a spherical triangle, the sines of the sides are proportional to the sines of their opposite angles.

Let ABC represent an oblique-angled spherical triangle. From any vertex, as C, draw the arc of a great circle, CB′, perpendicular to the opposite side. The two triangles ACB′ and BCB′ will be right-angled at B′.



From the triangle ACB′, we have, formula (2) Art. 74,

$$\sin CB' = \sin A \sin b.$$

From the triangle BCB′, we have

$$\sin CB' = \sin B \sin a.$$

Equating these values of sin CB′, we have

$$\sin A \sin b = \sin B \sin a;$$

from which results the proportion,

$$\sin a \;:\; \sin b \;::\; \sin A \;:\; \sin B. \quad \cdot \cdot \cdot \quad (1.)$$

In like manner, we may deduce

$$\sin a \;:\; \sin c \;::\; \sin A \;:\; \sin C, \quad \cdot \cdot \cdot \quad (2.)$$

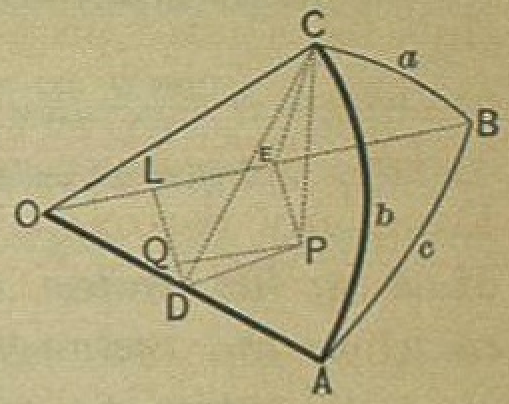
$$\sin b \;:\; \sin c \;::\; \sin B \;:\; \sin C. \quad \cdot \cdot \cdot \quad (3.)$$

That is, in any spherical triangle, *the sines of the sides are proportional to the sines of their opposite angles.*

Had the perpendicular fallen on the prolongation of AB, the same relation would have been found.

79. To find an expression for the cosine of any side of a spherical triangle.

Let ABC represent any spherical triangle, and O the centre of the sphere on which it is situated. Draw the radii OA, OB, and OC; from C draw CP perpendicular to the plane AOB; from P, the foot of this perpendicular, draw PD and PE respectively perpendicular to OA and OB; join CD and CE, these lines will be respectively perpendicular to OA and OB (B. VI., P. VI.), and the angles CDP and CEP will be equal to the angles A and B respectively. Draw DL and PQ, the one perpendicular, and the other parallel to OB. We then have



$$OE = \cos a, \qquad DC = \sin b, \qquad OD = \cos b.$$

We have from the figure,

$$OE = OL + QP. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (1.)$$

In the right-angled triangle OLD,

$$OL = OD \cos DOL = \cos b \cos c.$$

The right-angled triangle PQD has its sides respectively perpendicular to those of OLD; it is, therefore, similar to it, and the angle QDP is equal to c, and we have

$$QP = PD \sin QDP = PD \sin c. \quad \cdot \cdot \cdot \quad (2.)$$

The right-angled triangle CPD gives

$$PD = CD \cos CDP = \sin b \cos A;$$

substituting this value in (2), we have

$$QP = \sin b \sin c \cos A;$$

and now substituting these values of OE, OL, and QP, in (1), we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \quad \cdot \quad \cdot \quad (3.)$$

In the same way, we may deduce,

$$\cos b = \cos a \cos c + \sin a \sin c \cos B, \quad \cdot \quad \cdot \quad (4.)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \quad \cdot \quad \cdot \quad (5.)$$

That is, *the cosine of any side of a spherical triangle is equal to the rectangle of the cosines of the two other sides, plus the rectangle of the sines of these sides into the cosine of their included angle.*

80. To find an expression for the cosine of any angle of a spherical triangle.

If we represent the angles of the supplemental polar triangle of ABC, by A′, B′, and C′, and the sides by a', b', and c', we have (B. IX., P. VI.),

$$a = 180° - A', \qquad b = 180° - B', \qquad c = 180° - C',$$

$$A = 180° - a', \qquad B = 180° - b', \qquad C = 180° - c'.$$

Substituting these values in equation (3), of the preceding article, and recollecting that

$$\cos (180° - A') = - \cos A',$$

$$\sin (180° - B') = \sin B', \ \&c.,$$

we have

$$- \cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a' ;$$

or, changing the signs and omitting the primes (since the preceding result is true for any triangle),

$$\cos A = \sin B \sin C \cos a - \cos B \cos C. \quad \cdot \quad \cdot \quad (1.)$$

In the same way, we may deduce,

$$\cos B = \sin A \sin C \cos b - \cos A \cos C, \quad \cdot \quad \cdot \quad (2.)$$

$$\cos C = \sin A \sin B \cos c - \cos A \cos B. \quad \cdot \quad \cdot \quad (3.)$$

That is, *the cosine of any angle of a spherical triangle is equal to the rectangle of the sines of the two other angles into the cosine of their included side, minus the rectangle of the cosines of these angles.*

The formulas deduced in Arts. 79 and 80, for cos a, cos A, etc., are not convenient for use, as logarithms can not be applied to them; other formulas are, therefore, derived from them, to which logarithms may be applied.

81. To find an expression for the cosine of one half of any angle of a spherical triangle.

From equation (3), Art. 79, we deduce,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad . \quad . \quad . \quad . \quad (1.)$$

If we add this equation, member by member, to the number 1, and recollect that 1 + cos A, in the first member, is equal to $2 \cos^2 \frac{1}{2}A$ (Art. 66), and reduce, we have

$$2 \cos^2 \tfrac{1}{2}A = \frac{\sin b \sin c + \cos a - \cos b \cos c}{\sin b \sin c};$$

or, formula (C), Art. 65,

$$2 \cos^2 \tfrac{1}{2}A = \frac{\cos a - \cos (b + c)}{\sin b \sin c} \quad . \quad . \quad . \quad . \quad (2.)$$

And since, formula (N), Art. 67,

$$\cos a - \cos(b+c) = 2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(b+c-a),$$

equation (2) becomes, after dividing both members by 2,

$$\cos^2\tfrac{1}{2}A = \frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b\sin c}.$$

If in this we make

$$\tfrac{1}{2}(a+b+c) = \tfrac{1}{2}s;$$

whence, $$\tfrac{1}{2}(b+c-a) = \tfrac{1}{2}s - a,$$

and extract the square root of both members, we have

$$\cos\tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-a)}{\sin b\sin c}}. \quad . \quad . \quad . \quad (3.)$$

That is, *the cosine of one half of any angle of a spherical triangle is equal to the square root of the sine of one half of the sum of the three sides, into the sine of one half this sum minus the side opposite the angle, divided by the rectangle of the sines of the adjacent sides.*

If we subtract equation (1), of this article, member by member, from the number 1, and recollect that

$$1 - \cos A = 2\sin^2\tfrac{1}{2}A,$$

we find, after reduction,

$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s-b)\sin(\frac{1}{2}s-c)}{\sin b\sin c}}. \quad . \quad . \quad (4.)$$

Dividing equation (4) by equation (3), member by member, we obtain

$$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s-b)\sin(\frac{1}{2}s-c)}{\sin\frac{1}{2}s\sin(\frac{1}{2}s-a)}}. \quad . \quad . \quad (5.)$$

82. From the foregoing values of the functions of one half of any angle, may be deduced values of the functions of one half of any side of a spherical triangle.

Representing the angles and sides of the supplemental polar triangle of ABC as in Art. 80, we have

$$A = 180° - a', \qquad b = 180° - B', \qquad c = 180° - C',$$

$$\tfrac{1}{2}s = 270° - \tfrac{1}{2}(A' + B' + C'),$$

$$\tfrac{1}{2}s - a = 90° - \tfrac{1}{2}(B' + C' - A').$$

Substituting these values in (3), Art. 81, and reducing by the aid of the formulas in Table III., Art. 63, we find

$$\sin\tfrac{1}{2}a' = \sqrt{\frac{-\cos\frac{1}{2}(A'+B'+C')\cos\frac{1}{2}(B'+C'-A')}{\sin B'\sin C'}}.$$

Place $\qquad \tfrac{1}{2}(A' + B' + C') = \tfrac{1}{2}S;$

whence, $\qquad \tfrac{1}{2}(B' + C' - A') = \tfrac{1}{2}S - A'.$

Substituting and omitting the primes, we have

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos\frac{1}{2}S\cos(\frac{1}{2}S-A)}{\sin B\sin C}}. \quad . \quad . \quad (1.)$$

In a similar way, we may deduce from (4), Art. 81,

$$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(\frac{1}{2}S-B)\cos(\frac{1}{2}S-C)}{\sin B\sin C}}. \quad . \quad . \quad (2.)$$

and thence, $$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos\frac{1}{2}S\cos(\frac{1}{2}S-A)}{\cos(\frac{1}{2}S-B)\cos(\frac{1}{2}S-C)}}. \quad . \quad (3.)$$